AC RC CIRCUITS

A useful introduction to **AC circuits** can be developed using only **resistance** R and **capacitance** C. Picture an RC circuit driven by a sinusoidal voltage

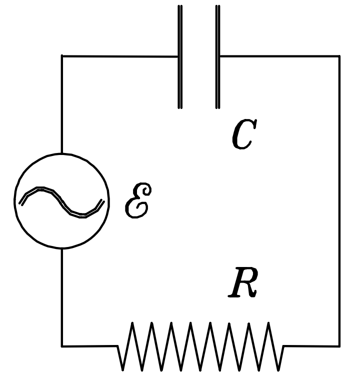
$$\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t) = \Re\, e^{i\omega t}$$

where $\Re$ signifies "the real part of" a complex quantity like $e^{i\theta} = \cos\theta + i\sin\theta$. The imaginary part is written (*e.g.*) $\Im\, e^{i\theta} = \sin\theta$.

(Physical quantities like current or voltage don't actually have a measurable imaginary part, of course.)

The voltage *amplitude* $\mathcal{E}_0$ is taken to be pure real.

An RC circuit driven by an AC voltage:



Kirchhoff's rule $\sum_i \Delta V_i = 0$ gives

$$\mathcal{E} - \frac{Q}{C} - R\frac{dQ}{dt} = 0 \, . \tag{1}$$

The only plausible "steady-state" motion is for Q to oscillate at the same frequency as the driving voltage. We express this expectation as a **trial solution**:

$$Q(t) = Q_0 e^{i\omega t} \, . \tag{2}$$

Let's see if this trial solution (2) "works" [satisfies the differential equation]. The complex exponential form is easy to differentiate: each time derivative of $Q(t)$ just "pulls down" another factor of $i\omega$. Thus

$$\mathcal{E}_0 e^{i\omega t} - \frac{1}{C} Q_0 e^{i\omega t} - i\omega R Q_0 e^{i\omega t} = 0 \,, \tag{3}$$

from which we can remove the common factor $e^{i\omega t}$ and do a little algebra to obtain

$$Q_0 = \frac{\mathcal{E}_0/R}{1/RC + i\omega} = \frac{\mathcal{E}_0/R}{\lambda + i\omega} \tag{4}$$

where

$$\lambda \equiv \frac{1}{RC} \equiv \frac{1}{\tau} \,. \tag{5}$$

Now, the charge on a capacitor can't be measured directly. What we want to know is the *current* $I \equiv \dot{Q}$. Since the entire time dependence of Q is in the factor $e^{i\omega t}$, we have trivially

$$I(t) = i\omega Q(t) = I_0 e^{i\omega t} \tag{6}$$

where

$$I_0 = i\omega Q_0 = \frac{i\omega \mathcal{E}_0/R}{\lambda + i\omega} = \frac{\mathcal{E}_0/R}{1 - i\lambda/\omega} = \frac{\mathcal{E}_0}{R - i/\omega C}\,. \tag{7}$$

Since $\mathcal{E}$, Q and I all have the same time dependence except for differences of *phase* encoded in the complex amplitudes Q_0 and I_0, we can think in terms of an *effective resistance* R_{eff} such that

$$\mathcal{E} = I R_{\text{eff}} \qquad \text{or} \qquad R_{\text{eff}} = \frac{\mathcal{E}_0}{I_0}\,. \tag{8}$$

With a little more algebra we can write the effective resistance in the form

$$R_{\text{eff}} = R - iX_C \tag{9}$$

where

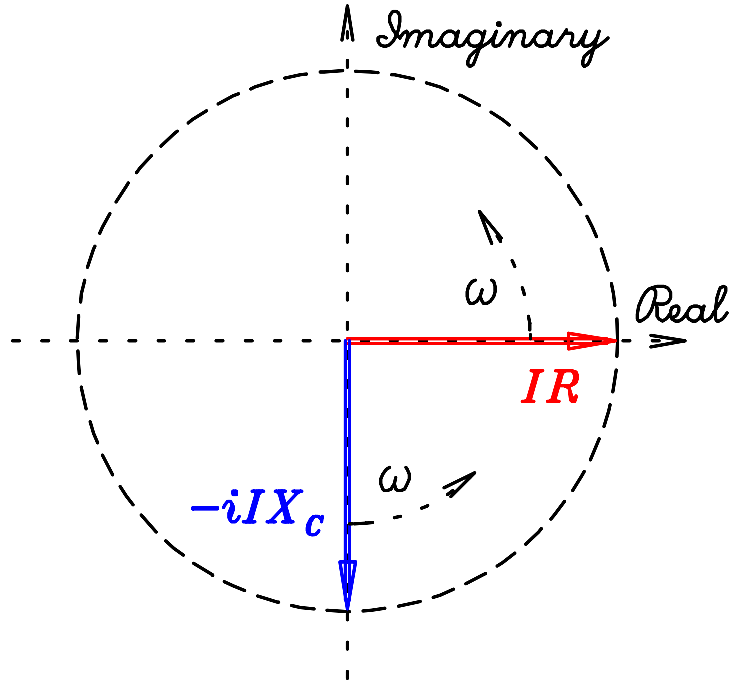
$$X_C \equiv \frac{1}{\omega C} \tag{10}$$

is the *capacitive reactance* of the circuit. This is a quantity that “acts like” (and has the units of) a *resistance* — just like R, the first term in R_{eff}.

The *current* through the circuit cannot be different in different places (due to charge conservation) and follows the time dependence of the driving voltage but (because R_{eff} is generally complex) is not generally *in phase* with it, nor with the *voltage drop* across C:

$$\begin{aligned} -\Delta\mathcal{E}_R &= IR\,, \qquad \text{but} \\ -\Delta\mathcal{E}_C &= -iIX_C\,. \end{aligned} \tag{11}$$

The **Phase Circle** shows the voltage drops in "complex phase space" as **vectors** that *rotate* at a constant frequency ω.



> The voltage across the *capacitor* *lags behind* that across the resistor by an angle of $\pi/2$.

At any instant the actual, *measurable* value of any voltage is just its *real* part — *i.e.* the projection of its complex vector onto the real axis.

Power

From the point of view of the power supply,* the circuit is a "black box" that "resists" the applied voltage with a weird "back $\mathcal{EMF}$" ($\mathcal{E}_{\text{back}}$) given by R_{eff} times the current I — *i.e.* by the sum of both terms in Eq. (11) or the sum of the two vectors in the *Phase Circle*.

*Please forgive my anthropomorphization of circuit elements; these metaphors help me remember their "behaviour".

The *power* dissipated in the circuit is the product of the real part of the applied voltage[†] and the real part of the resultant current[‡]

$$\begin{aligned} P(t) &= \Re\mathcal{E} \times \Re I = \Re\left(\mathcal{E}_0 e^{i\omega t}\right)\Re\left(I_0 e^{i\omega t}\right) \\ &= \mathcal{E}_0^2 \Re\left(\frac{1}{R_{\text{eff}}}\right)\cos^2(\omega t)\ . \end{aligned} \tag{12}$$

which oscillates at a frequency 2ω between zero and its maximum value

$$P_{\text{max}} = \mathcal{E}_0^2 \Re\left(\frac{1}{R_{\text{eff}}}\right) \tag{13}$$

[†]The imaginary voltage component doesn't generate any power.

[‡]Neither does the imaginary part of the current.

so that the *average* power drain is[§]

$$\langle P \rangle = \frac{1}{2}\mathcal{E}_0^2 \left[\frac{R}{R^2 + X_C^2}\right] = \mathcal{E}_{\text{rms}}\, I_{\text{rms}} \cos\phi \qquad (14)$$

where $\mathcal{E}_{\text{rms}} = \mathcal{E}_0/\sqrt{2}$, I_{rms} is the root-mean-square current in the circuit,

$$\cos\phi = \frac{R}{Z} \qquad (15)$$

is the "*power factor*" of the RC circuit and

$$Z \equiv \sqrt{R^2 + X_C^2} \qquad (16)$$

is the *impedance* of the circuit.

[§] I have used $\dfrac{1}{x+iy} = \dfrac{x - iy}{x^2 + y^2}$ to obtain the real part of $1/R_{\text{eff}}$.

Expressing the average power dissipation in this form allows one to think of an AC RC circuit the same way as a DC RC circuit with the *power factor* as a sort of "fudge factor".

This all gets a lot more interesting when we add the "inertial" effects of an **inductance** to our circuit. Stay tuned.